Learning Enhancement Team

University of East Anglia
STUDENT SUPPORT SERVICE

Steps into Calculus

Stationary Points of Functions of Two Variables

This guide explains how to find and classify stationary points for functions of more than one variable.

Introduction

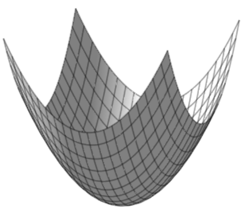
The study guides: *Stationary Points* and *Finding Stationary Points* explore stationary points of functions of a single variable. It would be useful to read these two guides before continuing on to familiarise yourself with the idea of a stationary point and how to find them. This study guide extends this basic idea to functions of two variables x and y. In other words this guide is concerned with finding stationary points of:

$$z = f(x,y)$$

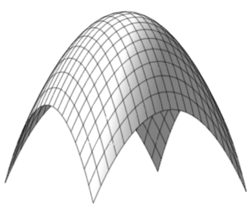
Stationary points of such functions occur when the function is neither changing as x changes nor changing as y changes.

Types of stationary points

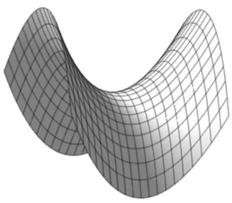
Just as for functions of one variable, there are three types of stationary points for functions of two variables: **minimums**, **maximums** and **saddle points**.



Minimum



Maximum



Saddle

You can think of a minimum as the bottom of a bowl, a maximum as the top of a hill and a saddle point as the place on a horses saddle where you can balance a marble without it rolling off.

Locating stationary points

The method of finding stationary points of multivariable functions is based on the idea that:

> The rate of change of a function is zero at a stationary point.

Finding the rates of change of these functions requires partial differentiation (see study guide: *Basics of Partial Differentiation*). There is a complication here as $z = f(x,y)$ is a function of both *x* and *y*, you can find its partial derivative with respect to either *x* or *y*:

$\frac{\partial z}{\partial x}$	partial derivative of *z* with respect to *x*
$\frac{\partial z}{\partial y}$	partial derivative of *z* with respect to *y*

[A note on notation: this guide will use the notation above for partial derivatives but you should be aware that these can also be written as z_x and z_y or f_x and f_y respectively.]

Stationary points of the function $z = f(x,y)$ are located where these partial derivatives are zero **simultaneously**. So, for a stationary point of $z = f(x,y)$:

$$\frac{\partial z}{\partial x} = 0 \quad \text{and also} \quad \frac{\partial z}{\partial y} = 0$$

These conditions will result in two equations which must be solved **simultaneously**, one for the partial derivative with respect to *x* and one for the partial derivative with respect to *y* (see study guide: *Simultaneous Equations*). If these equations can be solved, the solutions will give the *x*- and *y*-coordinates of any stationary points. Indeed stationary points are often reported as coordinates (x, y). If these equations cannot be solved then the function will have no stationary points.

Example: Find the stationary point(s) of the function $z = x^2 + y^2 + 3x + 2y$.

You begin by finding the partial derivatives with respect to *x* and *y* and setting the results equal to 0:

$$\frac{\partial z}{\partial x} = 2x + 3 \quad \text{gives} \quad 2x + 3 = 0 \qquad \text{so} \qquad x = -\frac{3}{2}$$

$$\frac{\partial z}{\partial y} = 2y + 2 \quad \text{gives} \quad 2y + 2 = 0 \quad \text{so} \quad y = -1$$

And so there is one stationary point located at $\left(-\frac{3}{2}, -1\right)$. Substituting these values into the original function gives:

$$z = \left(-\frac{3}{2}\right)^2 + (-1)^2 + 3\left(-\frac{3}{2}\right) + 2(-1) = -\frac{13}{4}$$

Example: Find the stationary point(s) of the function $z = xe^{-x^2-y^2}$.

This function is a bit more complicated but the method is the same, you begin by finding the partial derivatives with respect to *x* and *y* and setting the results equal to 0. For *x*:

$$\frac{\partial z}{\partial x} = e^{-x^2-y^2} - 2x^2e^{-x^2-y^2} = \left(1 - 2x^2\right)e^{-x^2-y^2} = 0$$

As the exponential part can never be zero (see study guide: *Exponential Functions*) you have:

$$1 - 2x^2 = 0 \qquad \text{so} \qquad x = \pm\frac{1}{\sqrt{2}}$$

giving two values of *x* for the stationary points. For *y*:

$$\frac{\partial z}{\partial y} = -2xye^{-x^2-y^2} = 0$$

Similarly, as the exponential part can never be zero you have:

$$-2xy = 0$$

You have already seen that $x = \pm 1/\sqrt{2}$ and so *x* is not zero and so $y = 0$ for any stationary points. Combining this with the two values of *x* gives two stationary points for this function at $\left(\frac{1}{\sqrt{2}}, 0\right)$ and $\left(-\frac{1}{\sqrt{2}}, 0\right)$.

Finding out about the type of stationary point you have

For functions of one variable, the type (often called the **nature**) of the stationary point depends on the behaviour of the second derivative of that function at that point. The type of stationary point for functions of two variables also depends on the behaviour of the second derivatives at those points. The second partial derivatives are specifically:

$$\frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^2 z}{\partial y^2} \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} \quad \text{(which is equal to } \frac{\partial^2 z}{\partial y \partial x}\text{)}$$

In order to work out whether you have a maximum, minimum or saddle point you need to calculate the parameter Δ where:

$$\Delta = \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$$

at a particular stationary point.

(The parameter Δ is the "discriminant of the Hessian of *z*". If you want to know more about this, a **Learning Enhancement Tutor** will be glad to talk to you about this piece of mathematics.)

Specifically:

If $\Delta > 0$ **and** $\frac{\partial^2 z}{\partial x^2} > 0$ the stationary point is a **minimum**.

If $\Delta > 0$ **and** $\frac{\partial^2 z}{\partial x^2} < 0$ the stationary point is a **maximum**.

If $\Delta < 0$ the stationary point is a **saddle point**.

If $\Delta = 0$ then you need to investigate the point further.

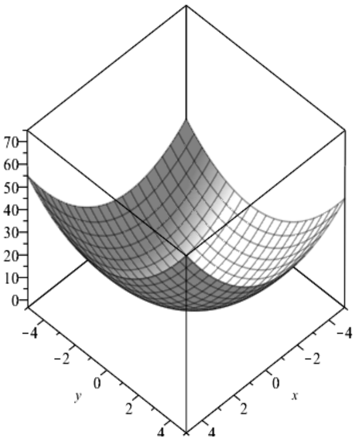
Example: What kind of the stationary point does the function $z = x^2 + y^2 + 3x + 2y$ have?

Firstly you should recognise that the stationary point of this function was found to be $\left(-\frac{3}{2}, -1\right)$ in the previous section. To find out what type of point you have you first need to find the second order partial derivatives. These are:

$$\frac{\partial^2 z}{\partial x^2} = 2 \qquad \frac{\partial^2 z}{\partial y^2} = 2 \qquad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = 0$$

These functions are not dependent on *x* or *y* and so the values can be directly substituted into Δ to give:

$$\Delta = \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 2 \cdot 2 - 0^2 = 4$$



So $\Delta > 0$ and $\frac{\partial^2 z}{\partial x^2} > 0$.

This tells you that $\left(-\frac{3}{2}, -1\right)$ is a minimum, as shown in the graph of $z = x^2 + y^2 + 3x + 2y$ to the left.

Example: Find the location and type of the stationary point of the function $z = x^2 - y^2$.

Firstly find the first derivatives and set these results equal to zero to find the location:

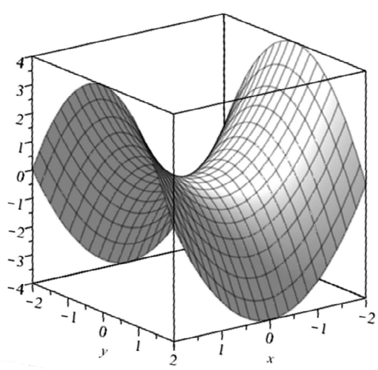
$$\frac{\partial z}{\partial x} = 2x \qquad \text{gives} \quad 2x = 0 \qquad \text{so} \qquad x = 0$$

$$\frac{\partial z}{\partial y} = -2y \qquad \text{gives} \quad -2y = 0 \qquad \text{so} \qquad y = 0$$

And so there is one stationary point located at the origin $(0,0)$. Now find the second derivatives to classify the stationary point:

$$\frac{\partial^2 z}{\partial x^2} = 2 \qquad \frac{\partial^2 z}{\partial y^2} = -2 \qquad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = 0$$

So $\Delta = \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 2 \cdot (-2) - 0^2 = -4$.



Therefore $(0,0)$ is a saddle point because $\Delta < 0$ as shown in the graph of $z = x^2 - y^2$ to the left. (This function is often referred to as the **saddle function**.)

Flowchart for finding and identifying stationary points

1. Find the **first derivatives** $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

2. Set these derivatives equal to zero and solve the resulting equations for *x* and *y*
   - → Substitute each pair of values of *x* and *y* back into the original function to find corresponding values of *z*

3. Find the **second derivatives** $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$

4. Substitute each pair of values of *x* and *y* into each of the second derivatives to find a number

5. Calculate Δ where $\Delta = \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$
   - → If $\Delta > 0$ **and** $\frac{\partial^2 z}{\partial x^2} > 0$ the stationary point is a **minimum**.
   - → If $\Delta > 0$ **and** $\frac{\partial^2 z}{\partial x^2} < 0$ the stationary point is a **maximum**.
   - → If $\Delta = 0$ further investigation is needed.
   - → If $\Delta < 0$ the stationary point is a **saddle point**.

Want to know more?

If you have any further questions about this topic you can make an appointment to see a **Learning Enhancement Tutor** in the **Student Support Service**, as well as speaking to your lecturer or adviser.

Call: 01603 592761

Ask: ask.let@uea.ac.uk

Click: https://portal.uea.ac.uk/student-support-service/learning-enhancement

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